14.7 Local Max/Min

Consider a surface z = f(x,y).

Some Terminology:

A **local maximum** occurs at (a,b) if f(a,b) is larger than all values "near" it (top of a hill).

A **local minimum** occurs at (a,b) if f(a,b) is smaller than all values "near" it (bottom of a valley).

A **critical point** is a point (a,b) where both

$$f_x(a,b) = 0 \quad \text{AND} \quad f_y(a,b) = 0$$

or a point where either partial does not exist.

Note: Local max/min occur at critical points!

Example:

Find the critical points of

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

Second Derivative Test

Let (a,b) be a critical point.

Find all second partials at (a,b)

($f_{xx}(a,b)$, $f_{yy}(a,b)$, $f_{xy}(a,b)$)

and compute

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

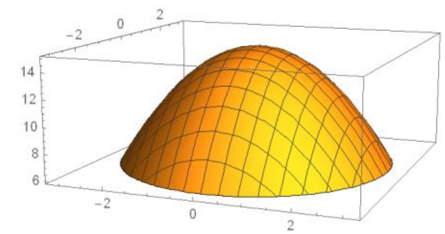
1. If **D > 0**, then the concavity is the same in all directions.
   (a) If $\mathbf{f_{xx} > 0}$, then it is concave up in all directions. So f(a,b) is a **local minimum.**
   (b) If $\mathbf{f_{xx} < 0}$, then it is concave down in all directions. So f(a,b) is a **local maximum.**

2. If **D < 0**, then the concavity changes in some direction.
   We say (a,b) is a **saddle point**.

3. If **D = 0**, the test is **inconclusive** (need a contour map)

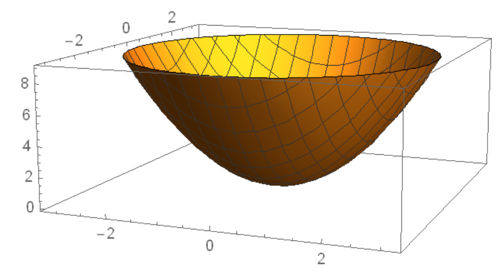
Quick Examples:

1. $f(x,y) = 15 - x^2 - y^2$,
only critical point
is (0,0).



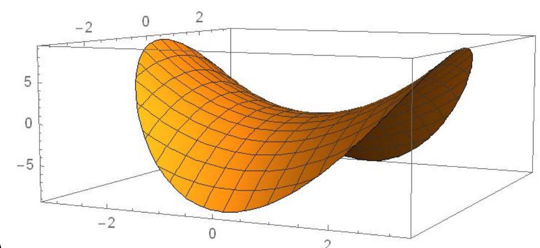
$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$,
$D = (-2)(-2)-(0)^2 = 4$

2. $f(x,y) = x^2 + y^2$,
only critical points
is (0,0).



$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$,
$D = (2)(2)-(0)^2 = 4$

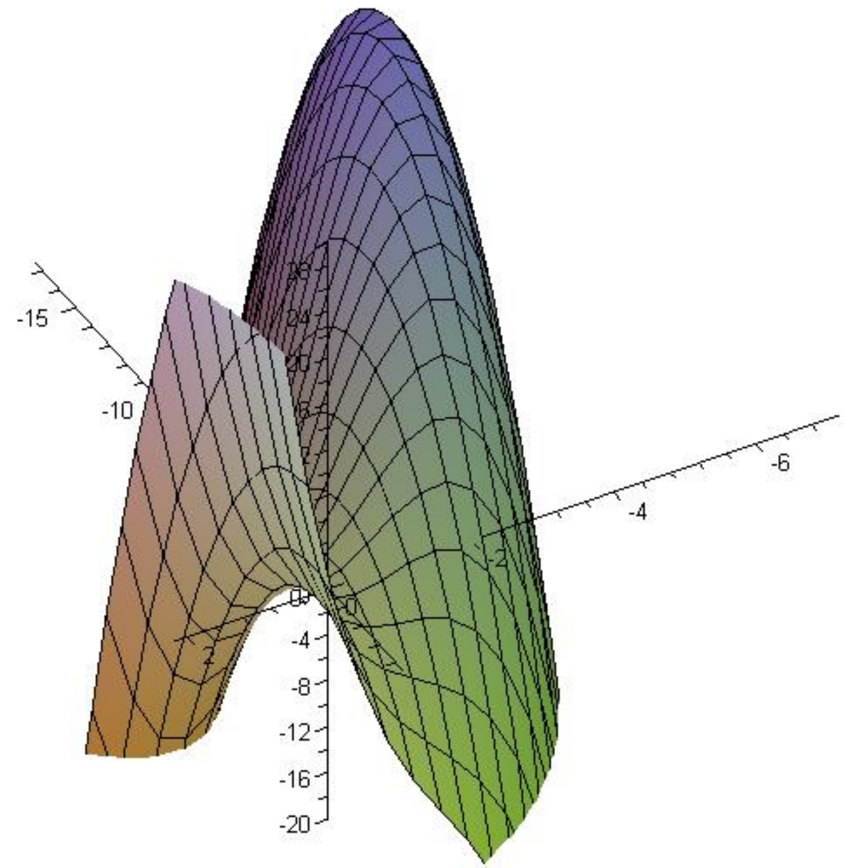
3. $f(x,y) = x^2 - y^2$
only critical points
is (0,0).



$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$,
$D = (2)(-2)-(0)^2 = -4$

Example: Find and classify all critical points for

$$f(x, y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$



14.7: Global Max/Min

Consider a surface f(x,y) over a particular region R on the xy-plane.

An **absolute/global maximum** over R is the largest z-value over R.

An **absolute/global minimum** over R is the smallest z-value over R.

Key fact (Extreme value theorem)
The absolute max/min must occur at either

1. A critical point, or
2. A boundary point.

Example: Let R be the triangular region in the xy-plane with corners at (0,-1), (0,1), and (2,-1). Over R, find the absolute max and min of

$$f(x,y) = \frac{1}{4}x + \frac{1}{2}y^2 - xy + 1$$

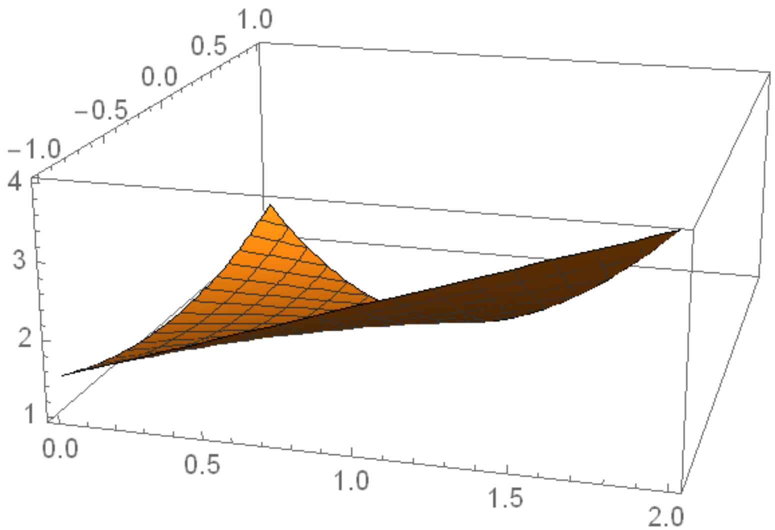
Step 1: Critical points inside region.

Step 2: Boundaries (the triangle has 3).
For each boundary, give equation find critical numbers of resulting one variable function.
Label critical numbers on each boundary.

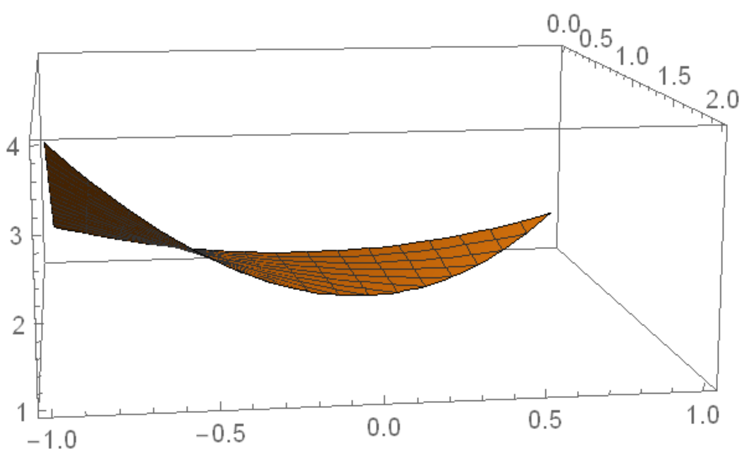
Step 3: Label corners.

Step 4: Evaluate the function at all points you found in steps 1, 2 and 3.

Biggest output = global max
Smallest output = global min



1.0
0.5
0.0
−0.5
−1.0
4
3
2
1
0.0
0.5
1.0
1.5
2.0



0.0
0.5
1.0
1.5
2.0
4
3
2
1
−1.0
−0.5
0.0
0.5
1.0

Homework hints:

In applied optimization problems,

(a) Identify what you are optimizing (objective)
(b) Label Everything.
(c) Identify any given facts (constraints)
(d) Use the constraints and labels to give a 2 variable function for the objective.

HW Examples:

1. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to (4,2,0).

Objective: Minimize **distance** from (x,y,z) points on the cone to the point (4,2,0).

2. Find the dimensions of the box with volume 1000 cm^3 that has minimum surface area.

Objective: Minimize **surface area**.